Chapter 2

The Structure of $\mathbb{R}$

A constant theme, which we begin in Chapter 2, is to show the similarities and dissimilarities between $\mathbb{Q}$ and $\mathbb{R}$. The triangle inequality and Corollary 2.1 are used extensively once we begin Chapter 3. The purpose of Exercise 2.1.11 is to have a problem which uses the triangle inequality and to introduce the student to an ε- type argument.

Some possibilities for student take home problems or projects are Exercises 4 or 8 of Section 2.2, Exercises 4 or 5 of Section 2.3, and Exercises 12 through 15 of Section 2.4. Exercise 2.4.15 will be difficult for many students simply because of the amount of structure; for example, you have a function whose range is a set of functions. Without some advice on notation, a student's solution to this problem could be very hard to follow.

2.1 Algebraic and Order Properties of $\mathbb{R}$

1. If $x+r=s\in\mathbb{Q}$, then $x=s-r\in\mathbb{Q}$ since $\mathbb{Q}$ is a field.
   If $rx=s\in\mathbb{Q}$, then $x=\dfrac{s}{r}\in\mathbb{Q}$ since $\mathbb{Q}$ is a field.

2. $-\sqrt{2}+\sqrt{2}=0$ and $\sqrt{2}\cdot\sqrt{2}=2$.

3. Suppose $x=\sqrt{2}+\sqrt{3}\in\mathbb{Q}$. Then $x^2\in\mathbb{Q}$ and $x^2=5+2\sqrt{6}$. Thus, $\sqrt{6}=\dfrac{x^2-5}{2}\in\mathbb{Q}$, a contradiction (see Exercise 1.1.1).

4. Let $0<a$ and $0<b$. Using Axiom 2.7 twice, $0=0+0<0+a<b+a$. Using Axiom 2.8 with $0<a$ and $b>0$, we have $0=0\cdot b<a\cdot b$. (Of course, this presupposes we know that $0\cdot b=0$ from the field axioms:

$0 \cdot b = (0+0) \cdot b = 0 \cdot b + 0 \cdot b$; now add the additive inverse, $-0 \cdot b$, to each side and use associativity.)

5. Suppose $1 < 0$. By Axiom 2.7, $0 = -1+1 < -1+0 = -1$. By Exercise 4, $0 < (-1)(-1) = 1$. Thus, $0 < 1$ and $1 < 0$, a contradiction to Axiom 2.6.

6.

$$\begin{aligned}
.33474747\ldots &= \frac{33}{100} + \frac{47}{10^4} + \frac{47}{10^6} + \frac{47}{10^8} + \cdots \\
&= \frac{33}{100} + \frac{47}{10^4}\left(1 + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^4 + \cdots\right) \\
&= \frac{33}{100} + \frac{47}{10^4}\frac{1}{1 - \frac{1}{100}} \\
&= \frac{33}{100} + \frac{47}{100}\frac{1}{99} \\
&= \frac{33(99) + 47}{9900} \\
&= \frac{3314}{9900} \\
&= \frac{1657}{4950}.
\end{aligned}$$

Your students may be familiar with the following technique. Let $x = .334747\ldots$. Then $100x = 33.4747\ldots$ and $10{,}000x = 3347.4747\ldots$. Subtracting we have $9900x = 3314$ or $x = \dfrac{3314}{9900}$.

7.
   1. If $a > 0$, then $|a| = a > 0$ and if $a < 0$, then $|a| = -a > 0$. So if $|a| = 0$, then $a = 0$. Of course, $|0| = 0$ by Definition 2.1.
   2. If $a \geq 0$, then $-a \leq 0$ and so $|-a| = -(-a) = a = |a|$.
      If $a < 0$, then $-a > 0$ and so $|-a| = -a = |a|$.

8. If $a \geq 0$, then $\sqrt{a^2} = a = |a|$.
   If $a < 0$, then $\sqrt{a^2} = -a = |a|$.

9. $a \leq b \Rightarrow a^2 = a \cdot a \leq a \cdot b$ (since $a \geq 0$) and $a \leq b \Rightarrow a \cdot b \leq b \cdot b = b^2$.
   Transitivity $\Rightarrow a^2 \leq b^2$.
   Let $a^2 \leq b^2$ and assume $a \neq b$. Then $b^2 - a^2 > 0$ or $0 < (b-a)(b+a)$.
   Since $b + a > 0$ by Exercise 4, $b - a > 0$ and so $b > a$.

10. From the proof of the triangle inequality, equality holds $\Leftrightarrow |ab| = ab \Leftrightarrow ab \geq 0$.

11. Let $\varepsilon = \frac{1}{2}|a-b|$. Let $U = (a-\varepsilon, a+\varepsilon)$ and $V = (b-\varepsilon, b+\varepsilon)$. Suppose $x \in U \cap V$. Then $|x-a| < \varepsilon$ and $|x-b| < \varepsilon$. So $|a-b| \leq |a-x| + |x-b| < 2\varepsilon = |a-b|$, a contradiction. So, $U \cap V = \emptyset$.

2.2 The Completeness Axiom

1. (a) $A = \{x \in \mathbb{R} : 0 < x^2 < 2\} = \{x \in \mathbb{R} : -\sqrt{2} < x < \sqrt{2}\} \setminus \{0\}$. So $\sup A = \sqrt{2}$ and $\inf A = -\sqrt{2}$.
   (b) $B = \{x \in \mathbb{R} : x^2 < 2\} = (-\sqrt{2}, \sqrt{2})$, So $\sup B = \sqrt{2}$ and $\inf B = -\sqrt{2}$.
   (c) $C = \{x \in \mathbb{R} : 0 < x \text{ and } x^2 > 2\} = (\sqrt{2}, \infty)$. So $\sup C = \infty$ and $\inf C = \sqrt{2}$.
   (d) $D = \{x \in \mathbb{R} : x^2 > 2\} = (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$. So $\sup D = \infty$ and $\inf D = -\infty$.

2. Let s_1 be the greatest element of S. Then $x \leq s_1\ \forall x \in S$, and so s_1 is an upper bound of S. Let γ be an upper bound of S. Then $x \leq \gamma$ $\forall x \in S$. In particular, $s_1 \leq \gamma$. Therefore, $s_1 = \sup S$.

3. Since $A \neq \emptyset$, $\inf A \leq \sup A$. Since $A \subset B$, $\sup B$ is an upper bound of B and hence of A. Since $\sup A$ is the least upper bound of A, $\sup A \leq \sup B$. Similarly, $\inf B$ is a lower bound of A and so $\inf B \leq \inf A$ since $\inf A$ is the greatest lower bound of A.

4. To show that $\inf(bS) = b \sup S$, let $\alpha = \sup S$. So $\alpha \in \mathbb{R}$ by the Completeness Axiom. <u>To show</u>: $b\alpha = \inf(bS)$. $\forall s \in S$, $s \leq \alpha \Rightarrow bs \geq b\alpha \Rightarrow b\alpha$ is a lower bound of bS. Let γ be a real lower bound of bS. <u>To show</u>: $\gamma \leq b\alpha$. $\forall s \in S$, $\gamma \leq bs \Rightarrow \frac{\gamma}{b} \geq s \Rightarrow \frac{\gamma}{b}$ is an upper bound of S. Since $\alpha = \sup S$, $\alpha \leq \frac{\gamma}{b}$ and so $b\alpha \geq \gamma$.
   To show $\sup(bS) = b\inf(S)$, let $\beta = \inf S$. So $\beta \in \mathbb{R}$ by Proposition 2.4. <u>To show:</u> $b\beta = \sup(bS)$. $\forall s \in S, \beta \leq s \Rightarrow b\beta \geq bs \Rightarrow b\beta$ is an upper bound of bS. Let γ be a real upper bound of bS. <u>To show:</u> $b\beta \leq \gamma$. $\forall s \in S, bs \leq \gamma \Rightarrow s \geq \dfrac{\gamma}{b} \Rightarrow \dfrac{\gamma}{b}$ is a lower bound of S. Since $\beta = \inf S$, $\dfrac{\gamma}{b} \leq \beta$ and so $\gamma \geq b\beta$.

5. Since S is bounded above, $-S$ is bounded below. Use the first part of Exercise 4 with $b = -1$.

6. To show $\sup(a+S) = a + \sup S$, $\alpha = \sup S \in \mathbb{R}$ by the Completeness Axiom. $\forall s \in S$, $s \leq \alpha \Rightarrow a + s \leq a + \alpha \Rightarrow a + \alpha$ is an upper bound of $a + S$. Let γ be a real upper bound of $a + S$. To show: $a + \alpha \leq \gamma$. $\forall s \in S$, $a + s \leq \gamma \Rightarrow s \leq \gamma - a \Rightarrow \gamma - a$ is an upper bound of S. Since $\alpha = \sup S$, $\alpha \leq \gamma - a$ or $a + \alpha \leq \gamma$. Therefore, $a + \alpha = \sup(a + S)$.

   To show $\inf(a + S) = a + \inf S$, $\beta = \inf S \in \mathbb{R}$ by Proposition 2.4. $\forall s \in S, \beta \leq s \Rightarrow a + \beta \leq a + s \Rightarrow a + \beta$ is a lower bound of $a + S$. Let γ be a real lower bound of $a + S$. To show: $\gamma \leq a + \beta$. $\forall s \in S, \gamma \leq a + s \Rightarrow \gamma - a \leq s \Rightarrow \gamma - a$ is a lower bound of S. Since $\beta = \inf S$, $\gamma - a \leq \beta$ or $\gamma \leq a + \beta$.

7. Let $\alpha = \sup A$, $\beta = \sup B$, $\eta = \max\{\alpha, \beta\}$. (Note that α, β, and η are extended real numbers; the instructor may wish to consider special cases such as $A = \emptyset$, A unbounded above, etc.)
   To show: $\eta = \sup(A \cup B)$. If $x \in A$, then $x \leq \alpha \leq \eta$ and if $x \in B$, then $x \leq \beta \leq \eta$. So, η is an upper bound of $A \cup B$. If γ is an upper bound of $A \cup B$, then γ is an upper bound of A and γ is an upper bound of B. So, $\gamma \geq \alpha$ and $\gamma \geq \beta$. So $\gamma \geq \eta$. Thus, $\eta = \sup(A \cup B)$.
   The second part is similar. Let $\alpha = \inf A, \beta = \inf B, \eta = \min\{\alpha, \beta\}$.
   To show: $\eta = \inf(A \cup B)$. If $x \in A$, then $x \geq \alpha \geq \eta$ and if $x \in B$, then $x \geq \beta \geq \eta$. So η is a lower bound of $A \cup B$. If γ is a lower bound of $A \cup B$, then γ is a lower bound of both A and B. Hence, $\gamma \leq \alpha$ and $\gamma \leq \beta \Rightarrow \gamma \leq \eta$ and so $\eta = \inf(A \cup B)$.

8. $\forall n \in \mathbb{N}$, let $p(n)$ be the statement: any subset of $\mathbb{R}$ with n elements contains a maximum element. Clearly, $p(1)$ is true. Assume $p(k)$ is true and let $S = \{x_1, \ldots, x_k, x_{k+1}\}$ be a subset of $\mathbb{R}$ with $k+1$ elements. Then $A = \{x_1, \ldots, x_k\}$ has a maximum element by the induction hypothesis. By Exercise 7, $\sup S = \max\{\max A, x_{k+1}\} \in S$. Therefore, S has a maximum element.
   For the minimum element, change the word maximum above to minimum (there are three of these changes) and use $\inf S = \min\{\min A, x_{k+1}\}$ in the next to last sentence.

9. Let $A = B = [-1, 0]$. Then $\alpha = \beta = 0$, but $\sup C = 1$. More simply, one could use $A = B = \{-1, 0\}$ for then $C = \{0, 1\}$.

2.3 The Rational Numbers are Dense in $\mathbb{R}$

1. $\beta = \inf S \in \mathbb{R}$. Let $\varepsilon > 0$. If $\beta + \varepsilon \leq s$ $\forall s \in S$, then $\beta + \varepsilon$ is a lower

bound of S, contradicting β being the greatest lower bound of S. So $\exists s_1 \in S$ with $s_1 < \beta + \varepsilon$.

2. Let m be the smallest element of $\{n \in \mathbb{N} : x < n\}$ ($\mathbb{N}$ is well-ordered and this set is nonempty by Theorem 2.1). Then $m - 1 \leq x < m$. For uniqueness, if m_1 and m_2 both satisfy this inequality with $m_1 > m_2$, then $x < m_2 \leq m_1 - 1$, a contradiction.

3. (a) $A = \left\{1 - \dfrac{1}{n} : n \in \mathbb{N}\right\} = \left\{0, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \ldots\right\}$ is clearly bounded above by 1. If $0 < x < 1$ with x an upper bound of A, then $\exists\, n \in \mathbb{N}$ such that $\dfrac{1}{n} < 1 - x$ or $x < 1 - \dfrac{1}{n}$, a contradiction. Therefore $\sup A = 1$ and $\inf A = 0$.

   (b) $\sup \mathbb{Q} = +\infty$ and $\inf \mathbb{Q} = -\infty$.

   (c) $C = \left\{n - \dfrac{1}{n} : n \in \mathbb{N}\right\} = \left\{0, \dfrac{3}{2}, \dfrac{8}{3}, \dfrac{15}{4}, \ldots\right\}$. $\sup C = +\infty$ and $\inf C = 0$.

   (d) $D = \{x \in \mathbb{Q} : x^2 < 2\} = \{x \in \mathbb{Q} : -\sqrt{2} < x < \sqrt{2}\}$. So $\sup D = \sqrt{2}$ and $\inf D = -\sqrt{2}$. (If $\sup D < \sqrt{2}$, since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists\, q \in \mathbb{Q}$ with $\sup D < q < \sqrt{2}$.)

4. We mimic the proof of Theorem 2.2 replacing $\dfrac{1}{n}$ with $\dfrac{\sqrt{2}}{n}$ which is, of course, irrational.

   Case 1. $0 < x$. Choose $n \in \mathbb{N}$ with $\dfrac{1}{n} < \dfrac{x}{\sqrt{2}}$. Then $0 < \dfrac{\sqrt{2}}{n} < x$. Note that this takes care of the case $y < 0 < x$.

   Case 2. $0 < x < y$. Choose $n \in \mathbb{N}$ with $0 < \dfrac{\sqrt{2}}{n} < y - x$. By Exercise 2 $\exists\, m \in \mathbb{N}$ such that $m - 1 \leq \dfrac{nx}{\sqrt{2}} < m$. So $\dfrac{m}{n} - \dfrac{1}{n} \leq \dfrac{x}{\sqrt{2}} < \dfrac{m}{n}$. Thus, $\dfrac{x}{\sqrt{2}} < \dfrac{m}{n} \leq \dfrac{x}{\sqrt{2}} + \dfrac{1}{n}$ and so $x < \dfrac{m\sqrt{2}}{n} \leq x + \dfrac{\sqrt{2}}{n} < x + (y - x) = y$ and $\dfrac{m\sqrt{2}}{n}$ is irrational.

   Cases 3 and 4. For $x < 0$ or $x < y < 0$, use the negative of the corresponding irrationals from Cases 1 and 2.

5. Let $\alpha = \inf A$, $\beta = \inf B$, and note that both α and β are in $\mathbb{R}$ by Proposition 2.4. <u>To show</u>: $\alpha + \beta = \inf(A + B)$. $\forall a \in A$ and $b \in B$,

$\alpha+\beta \le a+b \Rightarrow \alpha+\beta$ is a lower bound of $A+B$. Let γ be a real lower bound of $A+B$. To show: $\gamma \le \alpha+\beta$.

Argument 1. Fix $b_0 \in B$. $\forall a \in A$, $\gamma \le a+b_0 \Rightarrow \gamma - b_0 \le a \Rightarrow \gamma - b_0$ is a lower bound of A. Since $\alpha = \inf A$, $\gamma - b_0 \le \alpha$ or $\gamma - \alpha \le b_0$. Since b_0 is an arbitrary element of B, $\gamma - \alpha \le b \; \forall \, b \in B$ and so $\gamma - \alpha$ is a lower bound of B. Since $\beta = \inf B$, $\gamma - \alpha \le \beta$ or $\gamma \le \alpha + \beta$.

Argument 2. Let $\varepsilon > 0$. By Proposition 2.5, $\exists \; a_0 \in A$ and $b_0 \in B$ such that $a_0 < \alpha + \frac{\varepsilon}{2}$ and $b_0 < \beta + \frac{\varepsilon}{2}$. Since γ is a lower bound of $A+B$, $\gamma \le a_0 + b_0 < \alpha + \beta + \varepsilon$. Since ε is an arbitrary positive number, $\gamma \le \alpha + \beta$.

6. Let $\alpha = \sup\{f(x) : x \in X\}$ and $\beta = \sup\{g(x) : x \in X\}$. Both α and β are in $\mathbb{R}$ by the Completeness Axiom. $\forall x \in X$, $f(x)+g(x) \le \alpha+\beta \Rightarrow \alpha+\beta$ is an upper bound of $\{f(x)+g(x) : x \in X\}$. Since the supremum of a set is the least upper bound of a set, $\sup\{f(x)+g(x) : x \in X\} \le \alpha+\beta$.
   The functions in Example 2.3 will serve as examples here. For the functions in part 2 of Example 2.3, $\sup\{f(x)+g(x) : x \in [0,2]\} = \sup\{0\} = 0$ and $\sup\{f(x) : x \in [0,2]\} = 1 = \sup\{g(x) : x \in [0,2]\}$.

7. First note that all infs and sups here are real numbers.

   (a) $\forall x \in X$, $\inf f(X) \le f(x) \le g(x) \Rightarrow \inf f(X)$ is a lower bound of $\{g(x) : x \in X\}$. Hence, $\inf f(X) \le \inf g(X)$. Similarly, $f(x) \le g(x) \le \sup g(X) \; \forall x \in X$ and so $\sup g(X)$ is an upper bound of $\{f(x) : x \in X\}$. Therefore, $\sup f(X) \le \sup g(X)$.

   (b) Fix $y_0 \in X$. $\forall x \in X, f(x) \le g(y_0) \Rightarrow g(y_0)$ is an upper bound of $\{f(x) : x \in X\}$. Hence, $\sup f(X) \le g(y_0)$. Since y_0 is an arbitrary element of X, $\sup f(X) \le g(y) \; \forall y \in X$. Thus $\sup f(X)$ is a lower bound of $\{g(y) : y \in X\}$, and so $\sup f(X) \le \inf g(X)$.

   (c) Let $f, g : [0,1] \to \mathbb{R}$ be defined by $f(x) = x^2$ and $g(x) = x$. Then $f(x) \le g(x) \; \forall x \in [0,1]$, $\sup f([0,1]) = 1$, and $\inf g([0,1]) = 0$.

2.4 Cardinality

1. Part 2 is the contrapositive of part 1. To show part 3, assume $B \neq \emptyset$ and write $A = \{x_1, \ldots, x_n\}$. Let n_1 be the first positive integer such that $x_{n_1} \in B$. If $B \neq \{x_{n_1}\}$, let n_2 be the first positive integer greater

than n_1 with $x_{n_2} \in B$. Continuing, realize B as $\{x_{n_1}, x_{n_2}, \ldots, x_{n_j}\}$ for some $n_j \leq n$.

2. $A \sim$ subset of B, which is finite by Proposition 2.8, part 3.

3. The result is clear if A or B is empty. For A and B both nonempty, let f be a bijection of $\{1, \ldots, n\}$ onto A and let g be a bijection of $\{1, \ldots, m\}$ onto B where $m, n \in \mathbb{N}$. Define h from $\{1, \ldots n, n+1, \ldots, n+m\}$ onto $A \cup B$ by

$$h(i) = \begin{cases} f(i) & \text{if } i \in \{1, \ldots, n\} \\ g(i-n) & \text{if } i \in \{n+1, \ldots, n+m\}. \end{cases}$$

Then h maps $\{1, \ldots, n+m\}$ onto $A \cup B$, and so $A \cup B$ is finite by part 1 of Proposition 2.9.
$\forall n \in \mathbb{N}$, let $p(n)$ be the statement: the union of n finite sets is finite. Clearly, $p(1)$ is true. Let $k \in \mathbb{N}$ and assume $p(k)$ is true. Let $A_1, A_2, \ldots, A_k, A_{k+1}$ be $k+1$ finite sets. Then $\bigcup_{i=1}^{k} A_i$ is finite by the induction hypothesis, and so $\bigcup_{i=1}^{k+1} A_i = \left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}$ is finite by the first part of this problem. Therefore, $p(k+1)$ is true. By induction, $p(n)$ is true $\forall\, n \in \mathbb{N}$.

4. Since X is infinite, $X \setminus \{x\}$ is infinite by Exercise 3. By Theorem 2.3, $\exists$ a sequence of distinct points $(x_n)_{n=1}^{\infty}$ in $X \setminus \{x\}$. Map $x \rightarrow x_1, x_n \rightarrow x_{n+1}$ $\forall n$, $y \rightarrow y$ otherwise. This map is a bijection of X onto $X \setminus \{x\}$.

5. Let $(x_n)_{n=1}^{\infty}$ be a sequence of distinct points in $(0,1)$. Map $0 \rightarrow x_1$, $1 \rightarrow x_2$, $x_n \rightarrow x_{n+2}$ $\forall n$, $x \rightarrow x$ otherwise.

6. Proof of part 1. By Proposition 2.8, assume A is infinite. Then $\mathbb{N} \sim A$. Since $A \sim B$ and $\sim$ is transitive, $B \sim \mathbb{N}$. Hence, B is countably infinite. Part 2 is the contrapositive of part 1.

7. $A \sim$ subset of B, which is countable by Proposition 2.11, part 3.

8. The map $n \rightarrow n+m$ is a bijection of $\mathbb{N}$ onto $\mathbb{N} \setminus \{1, 2, \ldots, m\}$.

9. $A \times B = \bigcup_{a \in A} \{(a,b) : b \in B\}$ is the countable union of countable sets. Hence, $A \times B$ is countable by Theorem 2.4. Note that $\forall a \in A$, $\{(a,b) : b \in B\} \sim B$ by the map $(a,b) \rightarrow b$.

10. If $A \setminus B$ were countable, then $A = (A \setminus B) \cup B$ would be countable by Theorem 2.4.

11. Choose a rational in each member of $\mathfrak{A}$ (since $\mathbb{Q}$ is dense in $\mathbb{R}$). Since $\mathfrak{A}$ is pairwise disjoint, these rationals are distinct. Thus, $\mathfrak{A} \sim$ subset of $\mathbb{Q}$ and hence countable.

12. $\mathfrak{S} = \bigcup_{a \in \mathbb{Q}} \{(a, b) : b \in \mathbb{Q},\, b > a\}$ is countable by Theorem 2.4.

13. Argument 1. Assume A is countable and use a Cantor diagonalization argument. If f is a bijection of $\mathbb{N}$ onto A and $f(n) = (x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$ $\forall n \in \mathbb{N}$, define the sequence $(y_n)_{n \in \mathbb{N}}$ by $y_n = 1 - x_{n_n}$ $\forall n \in \mathbb{N}$. That is, $y_n = 1$ if $x_{n_n} = 0$ and $y_n = 0$ if $x_{n_n} = 1$. Then $(y_n)_{n \in \mathbb{N}} \in A \setminus$ range f, a contradiction.
    Argument 2. Every x in $[0, 1]$ has a binary representation, that is a representation as a sequence of 0's and 1's. Note that some x have 2 representations; for example, $\dfrac{1}{2} = \dfrac{0}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \dfrac{1}{2^4} + \cdots$ or in terms of sequences, $(1, 0, 0, 0, \ldots) = (0, 1, 1, 1, \ldots)$. In these cases let us agree to use the representation with a tail of 0's. $\forall x \in [0, 1]$, let $f(x)$ be the binary representation of x (f is well-defined since we have a fixed representation $\forall x$). Then f is a $1 - 1$ map of $[0, 1]$ into A (f is not onto A because of the double representation of some x). Since $[0, 1]$ is uncountable, A is uncountable by Proposition 2.12, part 2 .

14. Following the hint, either $x_0 \in A$ or $x_0 \notin A$. If $x_0 \in A$, then by definition of A, $x_0 \notin f(x_0) = A$, a contradiction. Therefore, $x_0 \notin A$. Again, by the definition of A, $x_0 \in f(x_0) = A$, a contradiction. Therefore, there does not exist a function from X onto $\mathcal{P}(X)$.

15. Suppose $F : [0, 1] \overset{\text{onto}}{\to} A$. Write $F(x)$ as F_x $\forall\, x \in [0, 1]$. Then $F_x : [0, 1] \to \mathbb{R}$ $\forall x \in [0, 1]$. Define $h \in A$ by

$$h(x) = \begin{cases} 0 & \text{if} \quad F_x(x) \neq 0 \\ 1 & \text{if} \quad F_x(x) = 0 \end{cases}$$

    $\forall x \in [0, 1]$. Since F maps $[0, 1]$ onto A, $h = F_{x_0}$ for some $x_0 \in [0, 1]$. But $h(x_0) \neq F_{x_0}(x_0)$, a contradiction. Therefore, no such F exists. (Remark. Since a $1 - 1$ function from $[0, 1]$ into A exists, the cardinal number of A is larger than the cardinal number of $\mathbb{R}$.)

For those instructors who do not like the use of the Axiom of Choice in the proofs of part 1 of Propositions 2.9 and 2.12, we offer an alternative argument in the case when A is countably infinite. The case when A is finite is done similarly.

Let A be countably infinite and let f be a function from A onto B. To show: B is countable. Since A is countably infinite, $\exists$ a bijection g from $\mathbb{N}$ onto A, and so $f \circ g$ is a function from $\mathbb{N}$ onto B. Define $h : B \to \mathbb{N}$ by $h(b) =$ the smallest positive integer n such that $(f \circ g)(n) = b \; \forall \; b \in B$. Equivalently, $h(b)$ is the smallest element of $g^{-1}(f^{-1}(\{b\}))$. Then h is well-defined since $\mathbb{N}$ is well-ordered and $f \circ g$ maps $\mathbb{N}$ onto B, and h is 1-1 since $f \circ g$ is a function. Therefore, B is equivalent to a subset of $\mathbb{N}$, which is countable by Proposition 2.11.